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逐步型 I 區間設限下之最適逐步加速試驗

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中文摘要

本研究中我們將考慮相同間隔時間 τ 之下 k 階段逐步應力加速壽命試驗的問題。在壓力改變的時間點 $i\tau, i=1,\ldots,k$ 上，部份存活的測試元件自試驗中移除，而在每個壓力區間內，我們只蒐集到測試元件故障的個數，並非眞正的元件壽命。這種資料蒐集方式，我們稱之爲逐步型 I 區間設限。本研究我們將以變異數最適性和 D 最適性這兩種準則來建立最佳的間隔時間 τ，並且將進行相關的模擬分析研究。

關鍵詞：加速壽命試驗；逐步設限；轉折點；變異數最適性； D 最適性；區間設限。

Abstract

We consider in this work k-step-stress accelerated test with equal duration step τ. Censoring is allowed at each change stress point $i\tau$, $i = 1,\ldots,k$. We will use a union of two methods of collecting type-I censored data, namely progressive censoring and interval censoring. The problem of choosing the optimum τ is addressed using the variance and the D-optimality criteria. Some simulation study will be investigated.

Keywords: Accelerated life testing; Progressive censoring; Change-point; Variance optimality; D-optimality; Interval censoring.

1 Introduction

Censoring is very common in life tests. It usually applies when exact lifetimes are known for only a portion of the products and the remainder of the lifetimes are known only to exceed certain values under a life test. The most common censoring schemes are type I censoring and type II censoring. If an experimenter desires to remove functioning units at points other than the final termination point of the life test, the above two censoring schemes will not be of use to the experimenter. The traditional censoring does not allow for units to be removed from test at the points other than the final termination point. This allowance may be desirable when a compromise between reduced time of experimentation and the observation of at least some extreme lifetimes is sought. This leads us into the area of progressive censoring. Statistical inferences on the parameters of failure time distributions under progressive censoring have been studied by several authors such as Cohen (1963), Wong (1993), and Balakrishnan and Aggarwala (2000). Note that in progressive censoring, the number of removals are all pre-fixed. However, in some practical situations, these numbers may occur at random. Yuen and Tse (1996) indicated that, for example, in some reliability experiments, an experimenter may decide that it is inappropriate or too dangerous to carry on the testing on some of the tested units even though these units have not failed. In these cases, the pattern of removal is random.

Accelerated life test (ALT) is often used for reliability analysis. In step-stress scheme, a test unit is subjected to successively higher levels of stress. A test unit starts at a specified low stress for a specified length of time. If it does not fail, stress on it is raised and held a specified time. The stress is thus increased step by step until the test unit fails. Generally all test units go through the same specified pattern of stress levels and test times. The simplest step-stress ALT uses only two stress levels and we call simple step-stress ALT. The statistical inferences in this step-stress ALT has been investigated by several authors such as Miller and Nelson (1983) and Nelson (1990).

In practice, it is often impossible continuously to observe the testing process, even with censoring. The test units might be able to be inspected intermittently. That is, we can only record whether a test unit fails in an interval instead of measuring failure time exactly. Hence, data of this type are called grouped data. In the literature, grouped data have been studied by many researchers such as Cheng and Chen (1988) and Aggarwala (2001).

2 Model and Assumptions

Let us consider the following k-level step-stress accelerated life-testing scheme with type I progressive group-censoring: n units are simultaneously placed on a life test at stress setting x_1, and run until time τ, at which point the number of failed units n_1 are counted and r_1 surviving units are removed from the test; starting from time τ, the $n-n_1-r_1$ non-removed surviving units are put to a different stress x_2 ($x_1 < x_2$) and run until time 2τ, at which point the number of failures n_2 are counted and r_2 surviving units are removed from the test, and so on. At time $k\tau$, the number of failed units n_k are counted and the remaining surviving $r_k = n - \sum_{i=1}^{k} n_i - \sum_{j=1}^{k-1} r_j$ units are all removed, thereby terminating the test.

For any stress, the failure time distribution of the test unit is an exponential distribution. At stress level x_i, the mean lifetime θ_i of a test unit is a log-linear function of stress. That is,

$$\ln \theta_i = \beta_0 + \beta_1 x_i, \quad i = 1, 2, \ldots, k. \tag{1}$$

Here the β_0 and β_1 (< 0) are unknown parameters and $x_1 < x_2 < \cdots < x_k$. Therefore, the mean lifetime of a test unit at lower stress is longer than at the higher stress. The log-linear function is a common choice for the life-stress relationship because it includes both the power-law and the Arrhenius-law as special cases. Furthermore, failures occur according to a cumulative exposure model.

From previous assumptions, the cumulative distribution function of a test unit under k-level step-stress test is $F(t) = F(s_{i-1} + t - (i-1)\tau; \theta_i)$, for $(i-1)\tau < t \leq i\tau$, where

$$F(t; \theta_i) = 1 - e^{-\frac{t}{\theta_i}}, \tag{2}$$

$s_0 = 0$, and $s_{i-1} = \frac{\theta_{i-1}}{\theta_{i-2}}(\tau + s_{i-2})$ is the solution of $F(s_{i-1}; \theta_i) = F(\tau + s_{i-2}; \theta_{i-1})$, $i = 2, 3, \ldots, k$.

3 Maximum Likelihood Estimation

Suppose a type I progressively group-censored sample is collected as described in Section 2, beginning with a random sample of n units with an exponential failure time distribution. Let n_i be the number of units known to have failed in the interval $((i-1)\tau, i\tau]$ and let r_i be the number of surviving units being withdrawn from the test at time $i\tau$, for $i = 1, 2, \ldots, k$. Then, we have the fact that

$$n_i | n_{i-1}, \ldots, n_1, r_{i-1}, \ldots, r_1 \sim binomial(m_i, F_i(\tau)),$$

and

$$r_i | n_i, \ldots, n_1, r_{i-1}, \ldots, r_1 \sim uniform(0, m_i - n_i),$$

where $m_i = n - \sum_{j=1}^{i-1} n_j - \sum_{j=1}^{i-1} r_j$ is the number of non-removed surviving units at the beginning of the ith stage, and $F_i(\tau) = \frac{F(i\tau) - F((i-1)\tau)}{1 - F((i-1)\tau)}$, for $i = 1, 2, \ldots, k$. The likelihood function is then

$$L \propto \prod_{i=1}^{k} \left(1 - e^{-\frac{\tau}{\theta_i}}\right)^{n_i} \left(e^{-\frac{\tau}{\theta_i}}\right)^{m_i - n_i}.$$

Substituting for θ_i the expression from (1) involving β_0 and β_1, the log-likelihood function can be written as

$$\ln L \propto \sum_{i=1}^{k} \left[n_i \ln \left(1 - e^{-\tau e^{-(\beta_0+\beta_1 x_i)}} \right) + (m_i - n_i) \left(-\tau e^{-(\beta_0+\beta_1 x_i)} \right) \right]. \tag{3}$$

Thus, the equations to be solved for the maximum likelihood estimates of β_0 and β_1 are

$$\sum_{i=1}^{k} \tau e^{-(\beta_0+\beta_1 x_i)} \left[m_i - n_i \left(1 - e^{-\tau e^{-(\beta_0+\beta_1 x_i)}} \right)^{-1} \right] = 0,$$

and

$$\sum_{i=1}^{k} \tau x_i e^{-(\beta_0+\beta_1 x_i)} \left[m_i - n_i \left(1 - e^{-\tau e^{-(\beta_0+\beta_1 x_i)}} \right)^{-1} \right] = 0.$$

Under some mild regularity conditions, any of several maximum likelihood large-sample procedure might be used to make inferences about β_0 and β_1. One possibility is to employ the asymptotic normal approximation to obtain confidence intervals for β_0 and β_1. We now derive Fisher's information matrix. From (3), we have

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \beta_0^2} &= \sum_{i=1}^{k} \tau e^{-(\beta_0+\beta_1 x_i)} \Bigg\{ -m_i + \\ &\qquad \frac{n_i}{1 - e^{-\tau e^{-(\beta_0+\beta_1 x_i)}}} \left[1 - \frac{\tau e^{-(\beta_0+\beta_1 x_i)} e^{-\tau e^{-(\beta_0+\beta_1 x_i)}}}{1 - e^{-\tau e^{-(\beta_0+\beta_1 x_i)}}} \right] \Bigg\}, \end{aligned} \tag{4}$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \beta_0 \beta_1} &= \sum_{i=1}^{k} \tau x_i e^{-(\beta_0+\beta_1 x_i)} \Bigg\{ -m_i + \\ &\qquad \frac{n_i}{1 - e^{-\tau e^{-(\beta_0+\beta_1 x_i)}}} \left[1 - \frac{\tau e^{-(\beta_0+\beta_1 x_i)} e^{-\tau e^{-(\beta_0+\beta_1 x_i)}}}{1 - e^{-\tau e^{-(\beta_0+\beta_1 x_i)}}} \right] \Bigg\}, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \beta_1^2} &= \sum_{i=1}^{k} \tau x_i^2 e^{-(\beta_0+\beta_1 x_i)} \Bigg\{ -m_i + \\ &\qquad \frac{n_i}{1 - e^{-\tau e^{-(\beta_0+\beta_1 x_i)}}} \left[1 - \frac{\tau e^{-(\beta_0+\beta_1 x_i)} e^{-\tau e^{-(\beta_0+\beta_1 x_i)}}}{1 - e^{-\tau e^{-(\beta_0+\beta_1 x_i)}}} \right] \Bigg\}. \end{aligned} \tag{6}$$

To obtain Fisher's information, we need the expectations of (4), (5) and (6). To get these, let us compute the expectations of m_i, n_i and r_i, for $i = 1, 2, \ldots, k$. Since $m_{i+1} = m_i - n_i - r_i$, $i = 1, 2, \ldots, k-1$, we can show that

$$E(m_i) = \frac{nG_{i-1}(\tau)}{2^{i-1}}, \quad E(n_i) = \frac{nG_{i-1}(\tau)F_i(\tau)}{2^{i-1}}, \quad \text{and} \quad E(r_i) = \frac{nG_i(\tau)}{2^i},$$

where $G_i(\tau) = \prod_{j=1}^{i} [1 - F_j(\tau)]$, for $i = 1, 2, \ldots, k$. A detailed proof of these expectations can be found in Wu and Lin (2004). Hence, the Fisher's information is

$$\mathbf{I}(\beta_0, \beta_1) = n \begin{bmatrix} \sum_{i=1}^{k} D_i(\tau) & \sum_{i=1}^{k} D_i(\tau) x_i \\ \sum_{i=1}^{k} D_i(\tau) x_i & \sum_{i=1}^{k} D_i(\tau) x_i^2 \end{bmatrix},$$

where

$$D_i(\tau) = \frac{G_{i-1}(\tau)}{2^{i-1}} \frac{[f_i(\tau)]^2}{[1 - F_i(\tau)]\, F_i(\tau)}, \qquad i = 1, 2, \ldots, k,$$

and

$$f_i(\tau) = \tau e^{-(\beta_0 + \beta_1 x_i)} e^{-\tau e^{-(\beta_0 + \beta_1 x_i)}}, \qquad i = 1, 2, \ldots, k.$$

For a moderate sample size n, the maximum likelihood estimator $(\hat{\beta}_0, \hat{\beta}_1)'$ is approximately distributed as a bivariate normal with mean $(\beta_0, \beta_1)'$ and variance-covariance matrix

$$\mathbf{I}^{-1}(\beta_0, \beta_1) = \frac{\begin{bmatrix} \sum_{i=1}^{k} D_i(\tau) x_i^2 & -\sum_{i=1}^{k} D_i(\tau) x_i \\ -\sum_{i=1}^{k} D_i(\tau) x_i & \sum_{i=1}^{k} D_i(\tau) \end{bmatrix}}{n \left(\sum_{i=1}^{k} \sum_{j=1}^{k} D_i(\tau) D_j(\tau) (x_i - x_j)^2 \right)}.$$

Therefore, the approximate confidence intervals for β_0 and β_1 or the asymptotic joint confidence region for $(\beta_0, \beta_1)'$ can be easily obtained.

4 Optimal Length of the Inspection Interval

The main purpose of this paper is to study the choice of τ, length of the inspection interval, in a k-level step-stress accelerated life test with type I progressive group-censoring. We investigate two selection criteria which enable one to choose the optimal value of τ.

Variance Optimality: The mean of the failure time distribution is an important characteristic and indispensable in reliability analysis. In step-stress setting, we need to estimate the mean lifetime at the use-condition with maximum precision. We can use the asymptotic variance of the logarithm of mean lifetime at use-condition as the criterion for selecting the optimal length of the inspection interval.

Let θ_0 be the mean lifetime at use-condition. The asymptotic variance of the estimator of $\ln\theta_0$ is

$$\begin{aligned} AVar(\ln\hat{\theta}_0) &= [1, x_0]\mathbf{I}^{-1}(\beta_0, \beta_1)[1, x_0]' \\ &= \frac{\sum_{i=1}^{k} D_i(\tau)(x_i - x_0)^2}{n\sum_{i=1}^{k}\sum_{j=1}^{k} D_i(\tau)D_j(\tau)(x_i - x_j)^2}, \end{aligned}$$

where x_0 is the stress at use-condition. The criterion function is then defined by

$$\phi(\tau) = \frac{\sum_{i=1}^{k} D_i(\tau)(x_i - x_0)^2}{\sum_{i=1}^{k}\sum_{j=1}^{k} D_i(\tau)D_j(\tau)(x_i - x_j)^2}.$$

The variance optimal τ is obtained by minimizing $\phi(\tau)$.

D-optimality: Another optimal criterion is based on the determinant of the Fisher's information matrix. It is known that the determinant $|\mathbf{I}(\beta_0, \beta_1)|$ is proportional to the reciprocal of the volume of the asymptotic joint confidence region for $(\beta_0, \beta_1)'$ so that maximizing this determinant is equivalent to minimizing the volume of confidence region. Consequently, a larger value of the determinant of the Fisher's information matrix would correspond to higher joint precision of the estimators of β_0 and β_1. Motivated by this, the optimal length of the inspection interval is chosen so that

$$g(\tau) = \sum_{i=1}^{k}\sum_{j=1}^{k} D_i(\tau)D_j(\tau)(x_i - x_j)^2$$

is maximized. This is called the D-optimality criterion.

5 Simulation Results

We conducted a numerical study to investigate the optimal length of the inspection interval. Consider the equi-spaced stress levels $x_i = x_0 + id$, $i = 1, 2, \ldots, k$, where $d > 0$ is the amount of stress increased at each stage. It is easy to see that with this choice, the relation between mean lifetimes of the ith and the $(i+1)$th stages is $\theta_{i+1} = \rho\theta_i$, $i = 1, 2, \ldots, k-1$, where $0 < \rho < 1$. Let $\tilde{\tau}_V$ and $\tilde{\tau}_D$ be the optimal length of the inspection interval according to the variance optimality and D-optimality criteria, respectively. Table 1 presents $\tilde{\tau}_V$ and $\tilde{\tau}_D$ values for $k = 2, 3, 4$, when θ_1 equals $100, 200, 300, 400, 500$ and ρ equals $0.5, 0.6, 0.7, 0.8, 0.9$. The findings are summarized as follows.

1. For fixed θ_1 and ρ, both $\tilde{\tau}_V$ and $\tilde{\tau}_D$ decrease as k increases. This means that larger the number of stress levels is, it is desirable to have a short length of the inspection interval.

2. For $k = 3, 4$, the D-optimal length of the inspection interval $\tilde{\tau}_D$ is always smaller than the variance-optimal length of the inspection interval $\tilde{\tau}_V$. However, for the simple step-stress case, $\tilde{\tau}_D$ is larger than $\tilde{\tau}_V$ when ρ is equal to 0.8 and 0.9.

3. The behavior of the optimal τ's as a function of mean lifetime at the first stage is interesting. For given k and ρ, either of the ratios $\frac{\tilde{\tau}_V}{\theta_1}$ or $\frac{\tilde{\tau}_D}{\theta_1}$ is constant across values of θ_1. Therefore, when the optimal length of the inspection interval $\tilde{\tau}_V$ or $\tilde{\tau}_D$ with respect to the mean lifetime θ_1 at the first stage is determined, it is easy to obtain that the optimal length of the inspection interval with respect to $c\theta_1$ is $c\tilde{\tau}_V$ or $c\tilde{\tau}_D$, where c is a known constant. This is true because θ_1 is a scale parameter of exponential distribution.

4. For fixed θ_1 and k, both $\tilde{\tau}_V$ and $\tilde{\tau}_D$ decrease as ρ decreases. That is, more severe the successive stages are (smaller the ρ), the optimal length of the inspection interval is shorter and, hence the experiment is stopped faster.

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Table 1: Optimal length of the inspection interval

θ_1	ρ	$k=2$		$k=3$		$k=4$	
		$\tilde{\tau}_V$	$\tilde{\tau}_D$	$\tilde{\tau}_V$	$\tilde{\tau}_D$	$\tilde{\tau}_V$	$\tilde{\tau}_D$
	0.5	75.4692	71.6758	52.1515	43.8127	45.7045	32.2313
	0.6	79.6970	77.7426	59.2608	52.2264	49.1017	39.2874
100	0.7	82.9269	82.6517	65.0987	59.4309	54.6046	46.7408
	0.8	85.4499	86.6897	69.8509	65.5947	60.0766	53.7304
	0.9	87.4585	90.0595	73.7291	70.8814	64.9527	60.0455
	0.5	150.9385	143.3516	104.3030	87.6254	91.4090	64.4625
	0.6	159.3940	155.4852	118.5216	104.4528	98.2033	78.5748
200	0.7	165.8538	165.3034	130.1974	118.8618	109.2092	93.4817
	0.8	170.8998	173.3793	139.7018	131.1894	120.1533	107.4609
	0.9	174.9170	180.1191	147.4581	141.7628	129.9055	120.0909
	0.5	226.4077	215.0274	156.4545	131.4381	137.1135	96.6938
	0.6	239.0910	233.2277	177.7824	156.6791	147.3050	117.8622
300	0.7	248.7807	247.9551	195.2961	178.2927	163.8138	140.2225
	0.8	256.3496	260.0690	209.5526	196.7840	180.2299	161.1913
	0.9	262.3755	270.1786	221.1872	212.6442	194.8582	180.1364
	0.5	301.8769	286.7032	208.6060	175.2507	182.8179	128.9250
	0.6	318.7880	310.9703	237.0432	208.9055	196.4066	157.1495
400	0.7	331.7075	330.6068	260.3947	237.7236	218.4184	186.9633
	0.8	341.7995	346.7586	279.4035	262.3787	240.3065	214.9217
	0.9	349.8340	360.2381	294.9163	283.5256	259.8110	240.1819
	0.5	377.3462	358.3790	260.7574	219.0634	228.5224	161.1563
	0.6	398.4850	388.7129	296.3040	261.1319	245.5083	196.4369
500	0.7	414.6344	413.2585	325.4934	297.1545	273.0230	233.7041
	0.8	427.2494	433.4483	349.2544	327.9734	300.3832	268.6522
	0.9	437.2925	450.2976	368.6453	354.4070	324.7637	300.2273